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SPECTRUM, ENTROPY, AND GEOMETRIC STRUCTURES FOR SMOOTH ACTIONS OF KAZHDAN GROUPS

BY

ROBERT J. ZIMMER*

Department of Mathematics, University of Chicago

Chicago, IL 60637, USA

ABSTRACT

We show that many natural classes of actions of discrete subgroups of semisimple Lie groups have discrete spectrum, i.e., are measurably conjugate to isometric actions.

1. Statement of Main Results

In this paper we describe a fundamental feature of the measure theoretic structure of certain smooth volume preserving actions of discrete subgroups of higher rank semisimple Lie groups on compact manifolds. This includes all actions in "low" dimensional manifolds, actions in which all elements act with zero entropy, and actions preserving a class of geometric structures. These results constitute a measure theoretic version of results conjectured to be true in [7,9] concerning the smooth structure of such actions.

Suppose G is a locally compact second countable group and (X, μ) is a standard measure space with $\mu(X) = 1$ on which G acts so as to preserve μ. A simple (but *a priori* very restricted) class of examples arises by considering a homomorphism $G \to K$ where K is a compact group, and letting X be a compact metric space on which K acts continuously. In this case, there is a K-invariant (and hence G-invariant) topological distance function on X. We say that an action is measurably isometric if it is measurably conjugate to such an action. If the action is measurably isometric and ergodic, then (up to measurable conjugacy) we may take $X = K/K_0$ where $K_0 \subset K$ is a closed subgroup. Thus, the measurably isometric ergodic actions (and via an ergodic decomposition any measurably

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isometric action) can be completely described, up to measurable conjugacy, in purely algebraic terms. If the compact group K is actually profinite, we say that the action is measurably profinite.

Returning to the general case, we let π be the unitary representation of G on $L^2(X, \mu)$ induced from the action. Following standard (but not entirely satisfactory) terminology, we say that the action of G on X has discrete spectrum if $L^2(X)$ is the direct sum of finite dimensional G-invariant subspaces. If the action is measurably isometric, then the Peter-Weyl theorem implies that the action has discrete spectrum, and it is a classical result of Mackey [1] (generalizing the corresponding result for integer actions of Halmos and von Neumann) that the converse is also true. In other words, one can characterise the spatial condition of being measurably isometric by the purely spectral condition of having discrete spectrum. An action is called weakly mixing if there are no finite dimensional G-invariant subspaces in $L^2(X)$ except $\mathbf{C}$. In the case of integer actions this is, of course, equivalent to the non-existence of an eigenvector for $\pi(g)$, $g \in \mathbf{Z}$, which is in turn equivalent to the classical definition of weak mixing via the limits of certain partial sums.

We also recall that for each $g \in G$, there is the (Kolmogorov-Sinai) entropy $h(g) \in [0, \infty]$. We say that the action has zero entropy if $h(g) = 0$ for all $g \in G$. Actions with discrete spectrum have zero entropy, but for the integers there is a vast array of zero-entropy weakly mixing actions. In fact, from a purely measure theoretic viewpoint, by virtue of Ornstein's theory of Bernoulli shifts, the complication in classifying integer actions arises at the zero-entropy level (or more precisely, in studying extensions with zero relative entropy). One of our main results will be the assertion that for smooth actions on compact manifolds of certain discrete subgroups of Lie groups, zero entropy is in fact equivalent to discrete spectrum.

We now give a precise description of the groups we will be considering and of the geometric aspects of our results. Let S be a finite subset of $\{\infty\} \cup \{\text{primes in } \mathbf{Z}\}$ and for each $p \in S$, let G_p be the set of $\mathbf{Q}_p$-points of a connected semisimple algebraic $\mathbf{Q}_p$-group, where $\mathbf{Q}_p$ is the field of p-adic numbers if p is a prime and $\mathbf{Q}_\infty = \mathbf{R}$. We assume the $\mathbf{Q}_p$-rank of every $\mathbf{Q}_p$-simple factor of G_p is at least 2. We let $G = \prod_{p \in S} G_p$, and shall abbreviate this situation by simply saying G is a semisimple group of higher rank. If $S = \{\infty\}$, we say G is real, in which case G is an almost connected semisimple Lie group with finite center. In fact, for real groups we shall also allow any connected semisimple group with finite center such that all simple factors have $\mathbf{R}$-rank at least 2, not just the algebraic ones. (Of course, any such group is algebraic modulo finite groups.)

If $S = \{p\}$, we simply say G is p-adic. Let Γ be a lattice in G, i.e., a discrete subgroup such that G/Γ has a finite G-invariant measure. The basic examples of such lattices are the S-arithmetic groups, and by a theorem of Margulis [2], [8] (which we will not be using) these are all the examples (in higher rank).

THEOREM 1.1: *Let G be a higher rank semisimple group and $\Gamma \subset G$ a lattice. Suppose Γ acts smoothly on a compact manifold preserving a volume density. Then the following are equivalent:*

(a) The Γ action has zero entropy.

(b) The Γ action is measurably isometric (or equivalently, has discrete spectrum).

(c) There is a Γ-invariant measurable Riemannian metric on M.

The implication (a) $\Rightarrow$ (c) is proven in [5]. (Actually [5] only considers the real case. However, these arguments work in general, using the results of [8, Chapter 10].) We have already remarked that (b) $\Rightarrow$ (a). The main argument of this paper is to show (c) $\Rightarrow$ (b). In [7] it is also shown that natural geometric hypotheses on the action imply condition (c). Namely, suppose $\dim M = n$ and $H \subset \mathrm{SL}(n, \mathbb{R})$ is a real algebraic subgroup. If Γ preserves an H-structure on M, then by [7] either there is a non-trivial Lie algebra homomorphism $\mathcal{G}_\infty \to \mathcal{H}$, or the Γ action has an invariant measurable Riemannian metric. Therefore, we deduce:

THEOREM 1.2: *Suppose every Lie algebra homomorphism $\mathcal{G}_\infty \to \mathcal{H}$ is trivial, and that Γ preserves an H-structure on M. Then the Γ-action is measurably isometric.*

We present two special cases of particular interest. Let $d(G) = \min\{d \mid$ there is a non-trivial Lie algebra representation $\mathcal{G}_\infty \to \mathcal{G}\ell(d, \mathbb{C})\}$.

COROLLARY 1.3: *If $n < d(G)$, then any volume preserving action of Γ on a compact n-manifold is measurably isometric.*

If G is p-adic, then $\mathcal{G}_\infty = (0)$. Hence:

COROLLARY 1.4: *If Γ is a lattice in a higher rank semisimple p-adic group, then any smooth volume preserving action on a compact manifold is measurably isometric. In particular, there are no such weakly mixing actions.*

If G is real, simple, has a $\mathbf{Q}$-structure with $\mathbf{Q}$-rank at least 2, and Γ is the arithmetic lattice $G_{\mathbf{Z}}$ (e.g. $G_{\mathbf{Z}} = \mathrm{SL}(n, \mathbf{Z})$ where $n \geq 3$), then every homomorphism of $G_{\mathbf{Z}}$ into a compact Lie group has finite image. Thus, every measurably isometric action is measurably profinite. For example:

COROLLARY 1.5: *Let $\Gamma = SL(n, \mathbf{Z})$ where $n \geq 3$. If M is a compact manifold of dimension d with $d < n$, then any smooth, volume preserving Γ-action on M is measurably profinite (and in particular is not weakly mixing).*

We have conjectured in [7, 9] that the conditions in Theorem 1.1 are in fact equivalent to the existence of a smooth Γ-invariant Riemannian metric, i.e., that the action is isometric, not just measurably isometric.

Our techniques in this paper also yield results for lattices in those $\mathbf{R}$-rank 1 groups with Kazhdan's property.

THEOREM 1.6: *Let Γ be a discrete Kazhdan group. Then any smooth volume preserving action of Γ on a compact surface is measurably isometric.*

Our proofs depend very strongly on the local analysis that is available by assuming the actions are smooth actions of manifolds. It would be very interesting to know whether or not conditions (a) and (b) in Theorem 1.1, for example, are equivalent outside the class of smooth actions.

As we indicated above, the proof of Theorem 1.1 is reduced to showing that the existence of a measurable invariant Riemannian metric implies that the action is measurably isometric. For an action of an arbitrary group this implication is false. A. Katok has constructed volume preserving diffeomorphisms of compact manifolds which have a measurable invariant Riemannian metric but which are weakly mixing. However, all the results stated above follow from the next theorem (when combined with the results of [5], [6], [7]). (In [6] it is shown that any Kazhdan group acting on a compact surface preserving a volume has an invariant measurable Riemannian metric.)

THEOREM 1.7: *Let Γ be a discrete Kazhdan group acting smoothly on a compact manifold preserving a smooth volume density. If there is a measurable Γ-invariant Riemannian metric then the action has discrete spectrum.*

The idea of the proof of Theorem 1.7 is as follows. Let $m \mapsto \omega(m) \in S^2(T^*M)$ be a measurable invariant Riemannian metric. Extend each $\omega(m)$ to a smooth metric ω_m defined in a neighborhood of $m \in M$. Fix $r > 0$, and for each m, let $f_m \in L^2(M)$ be the normalized characteristic function of the ball of radius r centered at m with respect to the metric ω_m. Define $F_r \in L^2(M \times M)$ by $F_r(m,x) = f_m(x)$. Fix a finite generating set $\Gamma_0 \subset \Gamma$ and let $\delta > 0$. By virtue of the invariance of $\omega(m)$, one can show that for r sufficiently small we have $\|\gamma \cdot F_r - F_r\| < \delta$ for $\gamma \in \Gamma_0$. In other words F_r is (Γ_0, δ)-invariant. By Kazhdan's property, if we choose δ sufficiently small this implies that F_r is close in $L^2(M \times M)$ to a Γ-invariant function. One can verify that this is not a constant function which yields the existence of some non-trivial finite dimensional Γ-invariant subspace in $L^2(M)$. One then needs to see that this general construction can be carried out so that these finite dimensional subspaces generate $L^2(M)$, an issue which presents further non-trivial complications.

2. General Properties of Measurable Actions of Kazhdan Groups

In this section we formulate some consequences of Kazhdan's property for a group acting on a measure space. Throughout this section Γ will be a discrete Kazhdan group and $\Gamma_0 \subset \Gamma$ a fixed finite symmetric generating set. If (N,ν) is a probability space and $B \subset N$ has positive measure, we let ν_B be the probability measure $\nu_B(A) = \nu(A \cap B)/\nu(B)$.

PROPOSITION 2.1: *Suppose $0 < q < 1$. Then there is some $\varepsilon > 0$ such that for any measure preserving Γ-action on a probability space (N,ν) and any set $D \subset N$ with $\nu_D(\gamma D \cap D) \geq 1-\varepsilon$ for all $\gamma \in \Gamma_0$, there is a Γ-invariant set $Y \subset N$ satisfying $\nu_Y(Y \cap D)$, $\nu_D(Y \cap D) > q$.*

For the proof, we first recall that Kazhdan's property asserts that there is some $\varepsilon > 0$ such that for any unit vector f in a Hilbert space on which Γ acts unitarily via π with no non-trivial invariant vectors, there is some $\gamma \in \Gamma_0$ such that $\|\pi(\gamma)f - f\| \geq \varepsilon$. (See [8].) This immediately admits the following reformulation.

LEMMA 2.2: *Given any $\delta > 0$, there is some $\varepsilon' > 0$ such that for any unitary representation (π, H) of Γ and any (ε', Γ_0)-invariant unit vector $f \in H$, there is a Γ-invariant unit vector $h \in H$ with $\|h - f\| < \delta$.*

Proof: Let ε be as above. If f is an $(\varepsilon/n, \Gamma_0)$-invariant unit vector, write $f = f_0 + f_1$ where $f_0 \in H_0$, the space of Γ-invariant vectors, and $f_1 \perp H_0$. Then f_1 is $(\varepsilon/n, \Gamma_0)$-invariant, and since $H_0^\perp$ has only trivial Γ-invariant vectors we have $\|\pi(\gamma)f_1 - f_1\| \geq \varepsilon\|f_1\|$ for some $\gamma \in \Gamma_0$. By ε/n-invariance, we deduce

$\varepsilon/n \geq \varepsilon\|f_1\|$, so $\|f_1\| \leq 1/n$. Therefore, $\|f_0\|^2 \geq 1-1/n^2$, $\|f-f_0\|^2 \leq 1/n^2$ and letting $h = f_0/\|f_0\|$, it is clear that for n sufficiently large we can take $\varepsilon' = \varepsilon/n$. □

Proof of Proposition 1.2: Given $0 < q < 1$, let $\delta > 0$ satisfy $q < (1-\delta)(1-\delta^{1/2})^2$ (and in particular $q < (1-\delta)$.) Let ε' be as in Lemma 2.2 for this choice of δ and set $\varepsilon = \varepsilon'/2$. If $A \subset N$, let f_A be the normalised (in L^2) characteristic function of A. Thus, $f_A \mid A = \nu(A)^{-1/2}$ and $f_A \mid A^c = 0$. Since Γ preserves ν we have

$$\|f_D - f_{\gamma D}\|^2 = \nu(\gamma D \bigtriangleup D)/\nu(D).$$

By the hypotheses on D we deduce that f_D is an (ε', Γ_0)-invariant unit vector. By Lemma 2.2, we can choose a unit vector $h \in L^2(N)$ that is Γ-invariant with $\|h - f_D\| < \delta$. Let

$$Y = \{x \in N | h(x) \geq (1-\delta^{1/2})\nu(D)^{-1/2}\}.$$

Then Y is Γ-invariant and on $D \cap Y^c$ we have

$$|h - f_D|^2 \geq \delta\nu(D)^{-1}.$$

Therefore,

$$\delta^2 \geq \|h - f_D\|^2 \geq \delta\nu(D)^{-1}\nu(D \cap Y^c),$$

and so

$$\frac{\nu(D \cap Y^c)}{\nu(D)} \leq \delta$$

and

$$\frac{\nu(D \cap Y)}{\nu(D)} \geq 1 - \delta > q.$$

On the other hand, we have $\int_Y h^2 \leq 1$, so

$$(1-\delta^{1/2})^2\nu(D)^{-1}\nu(Y) \leq 1.$$

Therefore

$$\nu(Y)^{-1} \geq (1-\delta^{1/2})^2\nu(D)^{-1}$$

and so

$$\frac{\nu(D \cap Y)}{\nu(Y)} \geq \frac{\nu(D \cap Y)}{\nu(D)}(1-\delta^{1/2})^2 \geq (1-\delta)(1-\delta^{1/2})^2 > q.$$

□

We will be applying Proposition 2.1 to the case in which $N = X \times M$ where X and M both have Γ-invariant probability measures and the Γ action on X is ergodic. To this end, we record the following elementary fact.

LEMMA 2.3: *Fix $0 < q < 1$, and $0 < \eta < 1$. Then there is some q' with $q < q' < 1$ such that for any $f : X \to [0,1]$ with $\int f(x) \geq q'$ we have $\mu(\{x | f(x) \geq q\}) \geq \eta$. (Here we are assuming (X, μ) is any probability space.)*

Proof: We have

$$q\mu(f^{-1}([0,q])) + \mu(f^{-1}([q,1])) \geq \int f \geq q'.$$

Thus, for $\alpha = \mu(f^{-1}[q,1])$, we have

$$q(1-\alpha) + \alpha \geq q',$$

so

$$\alpha \geq \frac{q'-q}{1-q}.$$

Thus, we need only choose $q' < 1$ such that $(q'-q)/(1-q) \geq \eta$. □

We now consider two Γ-spaces X and M with invariant probability measures. If $A \subset X \times M$ and $x \in X$, we let $A_x = \{m \in M | (x,m) \in A\}$.

COROLLARY 2.4: *Suppose (X, μ_0) and (M, μ) as above and that Γ acts ergodically on (X, μ_0). Let $N = (X \times M, \nu)$ where $\nu = \mu_0 \times \mu$. Given any $0 < q < 1$ and any $0 < \eta < 1$, there is some $\varepsilon > 0$ with the following property: For any set $D \subset X \times M$ satisfying*

(a) $\nu(\gamma D \cap D)/\nu(D) \geq 1 - \varepsilon$ for $\gamma \in \Gamma_0$, and

(b) $\mu(D_x)$ is independent of $x \in X$ (and hence equals $\nu(D)$),

there is a Γ-invariant set $Y \subset X \times M$ such that

$$\mu_0\left(\left\{x \Big| \frac{\mu(Y_x \cap D_x)}{\mu(Y_x)} \geq q, \quad \frac{\mu(Y_x \cap D_x)}{\mu(D_x)} \geq q\right\}\right) > \eta.$$

Proof: Given q and η choose q' as in Lemma 2.3. Apply Proposition 2.1 to q' to find ε. Then if $D \subset X \times M$ satisfies (a) and (b), we have a Γ-invariant $Y \subset X \times M$ such that

$$\frac{\nu(Y \cap D)}{\nu(Y)}, \frac{\nu(Y \cap D)}{\nu(D)} \geq q'.$$

We observe that $x \to \mu(Y_x)$ is Γ-invariant since Y is Γ-invariant, hence by ergodicity is constant, and that this constant must then be $\nu(Y)$. Thus

$$\int \frac{\mu(Y_x \cap D_x)}{\mu(Y_x)} = \frac{1}{\nu(Y)} \int \mu(Y_x \cap D_x) = \frac{\nu(Y \cap D)}{\nu(Y)}.$$

One has a similar assertion for $\mu(Y_x \cap D_x)/\mu(D_x)$ by hypotheses (b), and the result then follows from the conclusion of Lemma 2.3. □

3. Construction of Almost Invariant Sets

We now return to the situation in which (M, μ) is a compact n-dimensional manifold with a smooth Γ-invariant probability measure, and for which there is a measurable Γ-invariant Riemannian metric. We shall use a Taylor series estimate and Corollary 2.4 to produce small Γ-invariant measurable sets in $M \times M$ that are not too badly behaved with respect to balls in M.

We choose a point $p \in M$. For simplicity we may assume $p = 0 \in \mathbf{R}^n$ and that M contains a neighborhood of 0 in $\mathbf{R}^n$. By Moser's theorem [3] we may assume μ is the standard Lebesgue measure on this neighborhood of D. We then fix a smooth Riemannian metric on M which is the standard flat metric in a neighborhood of 0. We denote by $B(x, r)$ the ball of radius r centered at x with respect to this metric. We may assume that the smooth measure associated to this metric is μ. We let $J^k(M) \to M$ be the vector bundle of k-jets of $\mathbf{R}$-valued functions on M. The Riemannian metric on M determines a metric on these bundles, and if $\varphi \in \mathrm{Diff}_\mu(M)$ (the diffeomorphisms preserving μ) we denote by $\|\varphi\|_k$ the norm of the linear map φ induced on $C(M; J^kM)$, the Banach space of continuous sections of the bundle J^kM. Let $D_0 = \{\varphi \in \mathrm{Diff}_\mu(M) | \varphi(0) = 0\}$, $J^k(D_0)$ the group of k-jets at 0 of elements of D_0 and $p_k : D_0 \to J^k(D_0)$ the canonical projection. Let $x \mapsto h_x$ be a Borel map $M \to \mathrm{Diff}_\mu(M)$ such that

(i) $h_x(x) = 0$,

(ii) $\{\|h_x\|_2, \|h_x^{-1}\|_2 | x \in M\}$ is bounded.

The existence of such a map follows in a routine manner from compactness of M. Define $\alpha(\gamma, x) = h_{\gamma x} \circ \gamma \circ h_x^{-1}$, so that $\alpha : \Gamma \times M \to \mathrm{Diff}_\mu(M)$ is a Borel cocycle. Clearly $\alpha(\Gamma \times M) \subset D_0$, and if $\Gamma_0 \subset \Gamma$ is a finite set, then $\{\|\varphi\|_2 \,|\varphi \in \alpha(\Gamma_0 \times M)\}$ is a bounded set of real numbers.

The cocycle $p_2 \circ \alpha : \Gamma \times M \to J^2(D_0)$ is measurably equivalent to a cocycle into a compact subgroup of $J^2(D_0)$. This follows from the facts that:

(a) The existence of a measurable invariant Riemannian metric on TM is readily seen to be equivalent to the assertion that $p_1 \circ \alpha$ is equivalent to a cocycle taking all values in a compact subgroup K_1 of $J^1(D_0)$;

(b) $J^2(D_0) \cong J^1(D_0) \ltimes V$, where V is a vector group; and

(c) $K_1 \ltimes V$ is amenable, and hence any cocycle for a Kazhdan group taking values in $K_1 \ltimes V$ must be equivalent to one taking values in a compact subgroup. This last assertion is [8, Theorem 9.1.1] (cf. [7], Prop. 4.7).

We are not assuming that the smooth measure μ is ergodic. We fix a Γ-ergodic component (X, μ_0) of (M, μ) for which the cocycle $p_2 \circ \alpha$ is still equivalent to one into a compact subgroup K of $J^2(D_0)$. (This will be true for almost every ergodic component.) We may clearly assume that $K \subset O(n) \subset J^2(D_0)$, where we identify $O(n)$ with the 2-jets of orthogonal linear maps. We can in fact just let $X = M$, and view μ_0 as simply another Γ-invariant probability measure on M for which $p_2 \circ \alpha$ is equivalent to a cocycle into $K \subset J^2(D_0)$. We shall continue to denote M by X when we are considering it to be endowed with the measure μ_0. We note that we have no *a priori* smoothness properties of μ_0.

We now implement the equivalence of $p_2 \circ \alpha$ to a cocycle into K by a function $\overline{\psi}$. In other words, we choose a measurable $\overline{\psi} : X \to J^2(D_0)$, $x \mapsto \overline{\psi}_x$, such that

$$\beta(\gamma, x) = \overline{\psi}_{\gamma x}(p_2 \circ \alpha)(\gamma, x)\overline{\psi}_x^{-1} \in K \subset J^2(D_0).$$

Lift $\overline{\psi}$ to a measurable map $\psi : X \to D_0$. Then define the cocycle $\sigma : \Gamma \times X \to D_0$ by

$$\sigma(\gamma, x) = \psi_{\gamma x}\alpha(\gamma, x)\psi_x^{-1} = (\psi_{\gamma x} \circ h_{\gamma x}) \circ \gamma \circ (\psi_x \circ h_x)^{-1}.$$

Define $\lambda : X \to \mathbf{R}$ by

$$\lambda(x) = \max\{\|\psi_x \circ h_x\|_2, \quad \|(\psi_x \circ h_x)^{-1}\|_2\},$$

so that λ is a measurable function, and $\lambda(x) \geq 1$ (since these diffeomorphisms are all volume preserving).

THEOREM 3.1: *Let λ be as above and fix $1/2 < q < 1$, and $0 < \eta < 1$. Then for any $r > 0$ sufficiently small, there is a Γ-invariant measurable subset $Y^r \subset X \times M$ (depending upon r) and a measurable subset $X^r \subset X$ (depending upon r) such that:*

(1) $\mu_0(X^r) > \eta$.

(2) For all $x \in X^r$

$$\frac{\mu(Y_x^r \cap B(x, \lambda(x)r))}{\mu(Y_x)} \geq q.$$

(3) For all $x \in X^r$,

$$\frac{\mu(Y_x^r \cap B(x, \lambda(x)r))}{\mu(B(x, \lambda(x)r))} \geq c_0\lambda(x)^{-n}$$

where c_0 is a constant (independent of r and depending only on the original choice of smooth Riemannian metric on M), and $n = \dim M$.

Proof: Given q and η, choose ε as in Corollary 2.4 when the latter is applied to q and $\frac{\eta+1}{2}$. Now choose a set $X_1 \subset X$ such that for $X_0 = \bigcap_{\gamma\in\Gamma_0} \gamma X_1$ we have

$$\mu_0(X_0) > \max\{\frac{\eta+1}{2}, 1-\frac{\varepsilon}{2}\}$$

and such that λ is bounded on X_1. From the definition of σ it follows that there is a constant $\xi_0 \in \mathbf{R}$ such that $\|\sigma(\gamma, x)\|_2 \leq \xi_0$ for all $x \in X_0$ and all $\gamma \in \Gamma_0$.

For each $\varphi \in D_0$ such that $p_2 \circ \varphi \in O(n) \subset J^2(D_0)$, from the Taylor series expansion around 0 we have for x in a fixed neighborhood of 0 (independent of φ), that $|\varphi(x)| \leq |x| + \|\varphi\|_2 |x|^2$. Thus, for r sufficiently small, we have for all $\varphi \in \sigma(\Gamma_0 \times X_0)$ that

$$\varphi(B(0,r)) \subset B(0, r+\xi_0 r^2).$$

Assuming (as we may) that Lebesgue measure is suitably normalized, we have $\mu(B(0,r)) = r^n$. It follows that we can find r_1 so that for $r \leq r_1$,

$$\frac{\mu(B(0,r))}{\mu(B(0,r+\xi_0 r^2))} = \frac{r^n}{(r+\xi_0 r^2)^n} \geq 1 - \frac{\varepsilon}{4}.$$

Since any $\varphi \in \sigma(\Gamma_0 \times X_0)$ preserves μ, we also have

$$\frac{\mu(\varphi(B(0,r)))}{\mu(B(0,r+\xi_0 r^2))} \geq 1 - \frac{\varepsilon}{4}$$

for any such φ and $r \leq r_1$. Since both $B(0,r)$ and $\varphi(B(0,r))$ are contained in $B(0, r+\xi_0 r^2)$, we deduce

$$\frac{\mu(B(0,r) \cap \varphi(B(0,r)))}{\mu(B(0,r+\xi_0 r^2))} \geq 1 - \frac{\varepsilon}{2}$$

It follows that

$$\frac{\mu(B(0,r) \cap \varphi(B(0,r)))}{\mu(B(0,r))} \geq 1 - \frac{\varepsilon}{2}$$

as well.

We now use this to construct a set $D \subset X \times M$ satisfying the hypotheses of Corollary 2.4. Fix $r \leq r_1$. For $x \in X$, let $D_x = (\psi_x \circ h_x)^{-1} B(0,r) \subset M$. Then $\mu(D_x) = \mu(B(0,r))$ for all x. If $x \in X_0$ and $\gamma \in \Gamma_0$, then

$$\mu(\gamma D_x \cap D_{\gamma x}) \quad = \mu(\gamma(\psi_x \circ h_x)^{-1} B(0,r) \cap (\psi_{\gamma x} \circ h_{\gamma x})^{-1} B(0,r))$$

and since $\psi_{\gamma x} \circ h_{\gamma x}$ preserves μ this

$$\begin{aligned} &= \mu(\sigma(\gamma, x) B(0,r) \cap B(0,r)) \\ &\geq (1 - \frac{\varepsilon}{2}) \mu(B(0,r)) \end{aligned}$$

by the preceding paragraph. It follows that

$$\mu((\gamma D)_{\gamma x} \cap D_{\gamma x}) \geq (1 - \frac{\varepsilon}{2})\mu(B(0,r))$$

and therefore that

$$\begin{aligned}\nu(\gamma D \cap D) &= \int_X \mu((\gamma D)_x \cap D_x) d\mu_0(x) \\ &\geq (1 - \frac{\varepsilon}{2})\mu(B(0,r))\mu_0(X_0).\end{aligned}$$

Since

$$\nu(D) = \int \mu(D_x)\, d\mu_0(x) = \mu(B(0,x),$$

and $\mu_0(X_0) \geq 1 - \frac{\varepsilon}{2}$, we have

$$\frac{\nu(\gamma D \cap D)}{\nu(D)} \geq (1 - \frac{\varepsilon}{2})(1 - \frac{\varepsilon}{2}) \geq 1 - \varepsilon.$$

Applying Corollary 2.4 (and recalling that ε was chosen to work for the pair $(q, \frac{\eta+1}{2})$), it follows that for any $r \leq r_1$ we have a Γ-invariant set Y^r satisfying the conclusions of Corollary 2.4 for a set of x of μ_0-measure at least $1 + \frac{\eta}{2}$. Since $\mu_0(X_0) > \frac{1+\eta}{2}$ as well, we deduce that there is a set $X^r \subset X$ (in fact $X^r \subset X_0$) with $\mu_0(X^r) \geq \eta$ such that for $x \in X^r$, we have

(i) $\mu(Y_x^r \cap D_x)/\mu(Y_x^r) \geq q$,
(ii) $\mu(Y_x^r \cap D_x)/\mu(D_x) \geq q$.

From the definition of D_x and $\lambda(x)$, we see that $D_x \subset B(x, \lambda(x)r)$ and hence assertion (i) implies conclusion (2) of Theorem 3.1. It remains only to prove (3). By what we have just verified,

$$\begin{aligned}\frac{\mu(Y_x^r \cap B(x,\lambda(x)r))}{\mu(B(x,\lambda(x)r))} &\geq \frac{\mu(Y_x^r \cap D_x)}{\mu(D_x)} \frac{\mu(D_x)}{\mu(B(x,\lambda(x)r))} \\ &\geq q\frac{\mu(B(0,r))}{\mu(B(x,\lambda(x)r))}.\end{aligned}$$

The proof is then completed by the following remark.

LEMMA 3.2: *Let M be a compact Riemannian n-manifold with volume μ. Then there is some constant c_0 such that for all r sufficiently small, we have for all $m_1, m_2 \in M$ and all $\lambda \geq 1$ that*

$$\frac{\mu(B(m_1,r))}{\mu(B(m_2,\lambda r))} \geq c_0\lambda^{-n}.$$

Proof: This is clear in Euclidean space. By a local uniform comparison with a flat metric, it will be true locally. Compactness then ensures it is true on all M.

□

4. Proof of Theorem 1.7

The simple existence of a non-trivial Γ-invariant set Y in $X \times M$ (which is guaranteed by Theorem 3.1) is enough to show that there is some discrete spectrum in $L^2(M)$. We shall in fact prove the entire spectrum is discrete by making use of the additional information in Theorem 3.1.

We let (Z, ρ) be the maximal measurable factor of (M, μ) with discrete spectrum. That is, we have a Γ-space Z with invariant probability measure ρ, and a measure preserving Γ-map $M \to Z$ such that under the corresponding inclusion $L^2(Z) \hookrightarrow L^2(M)$, $L^2(Z)$ is the closure of the sum of all finite-dimensional Γ-invariant subspaces of $L^2(M)$. Such an (essentially unique) Z exists by [4, Thm. 7.1]. Our aim is to show $M = Z$.

For $z \in Z$, let M_z be the fiber over z and if $A \subset M$, let $A_z = A \cap M_z$. Let $\mu = \int^{\oplus} \mu_z d\rho(z)$ be the decomposition of μ over the fibers of $M \to Z$. We can write $Z = \bigcup Z_i$, $i \in \{\infty\} \cup \{n \in \mathbf{Z} | n \geq 0\}$ such that $\operatorname{card}(M_z) = i$ if $z \in Z_i$. From this it is easy to see that if $M \neq Z$, we can fix a set $A \subset M$ with $0 < \mu(A) < 1$ such that if $Z_0 = \{z \in Z | A_z \neq \emptyset\}$, then $\mu_z(A_z)$ is constant over $z \in Z_0$, and this constant is not 0 or 1. Define $\theta : M \to \mathbf{R}$ by defining $\theta_z = \theta | M_z$ to be $\theta_z = \chi_{A_z} - \mu_z(A_z)$. Therefore, there are constants $c, c' > 0$ and a decomposition $M = A \cup A' \cup A''$ such that $\theta | A = c$, $\theta | A' = -c'$, $\theta | A'' = 0$. Furthermore, $\theta \perp L^2(Z) \subset L^2(M)$. We fix such A, θ, c, c' throughout the remainder of the proof.

If $f \in L^2(X \times M)$ is Γ-invariant, let $T_f : L^2(M) \to L^2(X)$ be the corresponding integral operator. Then $T_f^* T_f$ is a compact self-adjoint operator in $L^2(M)$ whose eigenspaces for non-0 eigenvalues will be finite dimensional Γ-invariant subspaces. Since these are all contained in $L^2(Z)$ and $\theta \perp L^2(Z)$, it follows that we must have $\theta \in \ker(T_f^* T_f) = \ker(T_f)$ for any such f. To prove the theorem, it therefore suffices to construct such an f for which $\theta \notin \ker(T_f)$, thereby contradicting $M \neq Z$. We shall do this by taking f to be the characteristic function of a suitable set Y^r constructed in Theorem 3.1.

We first choose q and η in Theorem 3.1. First fix $1 > \eta > 1 - \mu_0(a)/8$. (By a suitable choice of ergodic component μ_0, we may assume $\mu_0(A) \neq 0$.) Now choose $X_1 \subset X$ such that $\mu_0(X_1) > \eta$ and there is $\Lambda \in \mathbf{R}$ with $\lambda(x) \leq \Lambda$ for all

$x \in X_1$. Then choose q so that

$$\max(\{c, c'\})\frac{(1-q)}{q} \leq cc_0\Lambda^{-n}/4$$

(where c, c' are as above and c_0 is as in Theorem 3.1). We claim that for this choice of η and q, by taking r sufficiently small $\theta \notin \ker T_f$, for $f = \chi_{Y^r}$.

By the Lebesgue density theorem, for μ-almost all $x \in A$ we have

$$\lim_{r\to 0} \frac{\mu(A \cap B(x,r))}{\mu(B(x,r))} = 1.$$

It follows that for almost all Γ-ergodic components of (M, μ) this same limit exists almost everywhere with respect to the ergodic component. Therefore, we may suppose that the ergodic component (X, μ_0) of (M, μ) was chosen originally so that for μ_0-almost all $x \in A$,

$$\lim_{r\to 0} \frac{\mu(A \cap B(x,r))}{\mu(B(x,r))} = 1.$$

Define, for $\omega < 1$,

$$A_{\omega,r} = \left\{x \in A \middle| \frac{\mu(A \cap B(x,s))}{\mu(B(x,s))} > \omega \quad \text{for all } 0 < s < r\right\}.$$

Choose $\omega < 1$ such that $(1-\omega)(c+c') \leq cc_0\Lambda^{-n}/2$. Then there is some R_ω such that if $r \leq R_\omega$, we have $\mu_0(A_{\omega,r}) \geq \mu_0(A)/2$.

Finally, choose $r > 0$ such that
(a) $\Lambda r < R_\omega$; and
(b) the conclusions of Theorem 3.1 hold with the above choices of η and q.
We then have $\mu_0(X^r) > 1 - \mu_0(A)/8$ (by choice of η), $\mu_0(X_1) > 1 - \mu_0(A)/8$ (by choice of X_1), and $\mu_0(A_{\omega,\Lambda r}) \geq \mu_0(A)/2$. It follows that $\mu_0(A_{\omega,\Lambda r} \cap X^r \cap X_1) > 0$. For ease of notation, with r thus chosen, we set $Y^r = Y$. We show $\theta \notin \ker T_f$ for $f = \chi_Y$. To do this it suffices to show that if $x \in A_{\omega,\Lambda r} \cap X^r \cap X_1$, then $(T_f\theta)(x) \neq 0$.

We have

$$\begin{aligned}(T_f\theta)(x) &= \int f(x,m)\theta(m)\,d\mu(m)\\ &= \int_{Y_x} \theta\,d\mu\\ &= I_1 + I_2\end{aligned}$$

where

$$I_1 = \int_{Y_x \cap B(x,\lambda(x)r)} \theta \quad \text{and} \quad I_2 = \int_{Y_x - Y_x \cap B(x,\lambda(x)r)} \theta.$$

Recalling that $\theta|A = c$, $\theta|A' = -c'$ and $\theta|A'' = 0$, we have

$$I_1 = c\mu(Y_x \cap B(x,\lambda(x)r) \cap A) - c'\mu(Y_x \cap B(x,\lambda(x)r) \cap A').$$

Since $x \in A_{\omega,\Lambda r}$ and $\lambda(x) < \Lambda$ (since $x \in X_1$), we have

$$\frac{\mu(B(x,\lambda(x)r) \cap A)}{\mu(B(x,\lambda(x)r))} > \omega,$$

and since $x \in X^r$, we have

$$\frac{\mu(Y_x \cap B(x,\lambda(x)r))}{\mu(B(x,\lambda(x)r))} \geq c_0\lambda(x)^{-n} \geq c_0\Lambda^{-n}.$$

Therefore

$$c\mu(Y_x \cap B(x,\lambda(x)r) \cap A) \geq c(c_0\Lambda^{-n} - [1-\omega])\mu(B(x,\lambda(x)r)).$$

For the second term in I_1 we observe

$$\frac{\mu(B(x,\lambda(x)r) \cap A')}{\mu(B(x,\lambda(x)r))} \leq 1 - \omega.$$

Therefore

$$-c'\mu(Y_x \cap B(x,\lambda(x)r) \cap A') \geq -c'(1-\omega)\mu(B(x,\lambda(x)r)).$$

Hence

$$I_1 \geq [cc_0\Lambda^{-n} - (1-\omega)(c+c')]\mu(B(x,\lambda(x)r)).$$

By the choice of ω, we deduce

$$I_1 \geq \frac{cc_0\Lambda^{-n}}{2}\mu(B(x,\lambda(x)r)).$$

To estimate I_2, observe that from Theorem 3.1 we have

$$\frac{\mu(Y_x \cap B(x,\lambda(x)r))}{\mu(Y_x)} \geq q,$$

so

$$\mu(Y_x - Y_x \cap B(x,\lambda(x)r)) \leq (1-q)\mu(Y_x).$$

Therefore,

$$|I_2| \leq \max(\{c, c'\})(1-q)\mu(Y_x).$$

We also have

$$\mu(Y_x) \leq \frac{\mu(Y_x \cap B(x, \lambda(x)r))}{q} \leq \frac{\mu(B(x, \lambda(x)r))}{q}.$$

Hence

$$|I_2| \leq \max(\{c, c'\})\frac{1-q}{q}\mu(B(x, \lambda(x)r)).$$

By the choice of q we then have

$$|I_2| \leq \frac{cc_0\Lambda^{-n}}{4}\mu(B(x, \lambda(x)r)).$$

This shows

$$I_1 + I_2 \geq \frac{cc_0\Lambda^{-n}}{4}\mu(B(x, \lambda(x)r)),$$

and this completes the proof of the theorem. □

5. Further Questions

Are there smooth volume preserving actions of Kazhdan groups on compact manifolds which have zero entropy but do not have discrete spectrum? For lattices in a higher rank of groups, Theorem 1.1 asserts that this is not the case. The additional information one has available in the higher rank case is of course deduced from superrigidity for cocycles.

Does every measure preserving zero entropy action on a compact metric space of a lattice in a higher rank group have discrete spectrum? More generally, is this true for Kazhdan groups?

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